Chapter Four:
Introduction to Dislocations
Imperfections
1. Point defects: vacancy atoms, substitutional and interstitial atoms (Ch. 7th)
2. Line defects: dislocations (Ch. 4th and 5th)
3. Area defects: grain boundaries (Ch. 6)

4.1 The Discrepancy Between the Theoretical and Observed Yield Stresses of Crystals
• e.g. a Mg single crystal; strain in tension => easily stretches out to a narrow ribbon much longer than the original crystal.
  => examine the surface of the deformed crystal, a series of fine steps have been formed (on basal plane (0002))
Fig. 4.1 Tensile stress-strain curve for a magnesium single crystal

Fig. 4.2 Slip lines on magnesium crystal
• The visible manifestations of a series of fine steps
• Sheared on a number of parallel planes.
• Crystallographic analyses: (0002) plane

**FIG. 4.3** (A) Magnified schematic view of slip lines (side view). (B) Magnified schematic view of slip lines (front view)
• Each atom of the upper plane slides over its neighbor, maximum called saddle point.
• Assume the radius of the atom is $a$.
• Shear strain $\gamma \approx \Delta x/x$
• Shear stress ($\tau$) = shear modulus ($\mu$) \times Shear strain ($\gamma$)
• Shear strain $\gamma$ at the saddle point $\approx a/2a = 0.5$
• For Mg: $\mu = 17.2$ GPa
  \[ \Rightarrow \tau \approx 9 \times 10^3 \text{ Mpa (theoretical)} \]

**Fig. 4.4 (A)** Initial position of the atoms on a slip plane.  
**(B)** The saddle point for the shear of one plane of atoms over another.  
**(C)** Final position of the atoms after shear by one atomic distance.
4.2 Dislocations

- Real crystals contain defects.
- Observable with TEM
  - dislocation lines on the slip plane
  - evidence: using a suitable etching solution
  - etch pits

**FIG. 4.5** (A) Schematic representation of an electron microscope photograph showing a section of a slip plane. (B) A three-dimensional view of the same slip plane section. (C) Termination of dislocations can also be revealed by etch pits.
Bright field image  Second grain (diffraction less)

FIG. 4.6 An electron micrograph of a foil removed from an aluminum specimen. Note the dislocations lying along a slip plane, in agreement with Fig. 4.5. (Photograph courtesy of E.J. Jenkins and J. Hren.)
1. Edge dislocation

- Extra planes: ab and cd
- Large distortion occurs around core region, decreasing intensity as one moves away
  => At large distances, the atoms tend to be arranged as they would be in a perfect crystal.
- This boundary of the additional plane: edge dislocation

![Diagram of edge dislocation](image)

**FIG. 4.7** An edge dislocation. (A) A perfect crystal. (B) When the crystal is sheared one atomic distance over part of the distance S–P, an edge dislocation is formed. (C) Three dimensional view of slip
3D Edge Dislocation

- **Dislocation line marks** sheared and unsheared area.
- **Definition**: a dislocation is a line that forms a boundary on a slip plane between slipped and non-slipped regions.

**FIG. 4.8** This three-dimensional view of a crystal containing an edge dislocation shows that the dislocation forms the boundary on the slip plane between a region that has been sheared and a region that has not been sheared.
• How the dislocation moves: (see 4.8 edge dislocation climb)
  \( c \rightarrow c' \), half plane \( x \rightarrow y \)
  \( \Rightarrow \) Continued \( \tau \rightarrow \) one atomic distance.
• This slip is slightly influenced by temp. (vs. jump)
The existence of dislocations was postulated much earlier (~25 years) than experimental techniques => Explain low yield strength of real crystals

The differentiation is purely arbitrary.

Positive edge dislocation

Negative edge dislocation
2. Screw dislocation

- ABCD slip plane
- The lattice planes of the crystal spiral the dislocation line DC.
- Starting point x, ending point z

**FIG. 4.10** Two representations of a screw dislocation. Notice that the planes in this dislocation spiral around the dislocation like a left-hand screw
• A model is readily constructed using a stack of filing cards.
• Cut and tape.
• A continuous spiral plane that goes entirely through the pack.
• Left-hand screw dislocation
• Thumb: direction of the screw axis
Fingers: direction of advance of the spiral plane toward thumb

**FIG. 4.11** Illustration of the construction of a model of a screw dislocation
2. Screw dislocation

Left-hand screw dislocation

Right-hand screw dislocation

FIG. 4.12 The ways that the four basic orientations of a dislocation move under the same applied stress: (A) Positive edge, (B) Negative edge, (C) Left-hand screw, and (D) Right-hand screw
3. Two Edge dislocations

- The basic nature of dislocations: cannot end inside a crystal

**FIG. 4.13** Dislocations can vary in direction. This shaded extra plane forms a dislocation with edge components $a$ and $b$
4. Edge and Screw Dislocations

**Fig. 4.14** A two-component dislocation composed of an edge and a screw component.

**Fig. 4.15** Atomic configuration corresponding to the dislocation of Fig. 4.14 viewed from above. Open-circle atoms are above the slip plane, dot atoms are below the slip plane.

**Fig. 4.16** A dislocation that changes its orientation from a screw to an edge as viewed from above looking down on its slip plane.
5. Dislocation Loop

**FIG. 4.17** A closed dislocation loop consisting of (a) positive edge, (b) right-hand screw, (c) negative edge, and (d) left-hand screw

**FIG. 4.18** A curved dislocation loop lying in a slip plane
4.3 The Burgers Vector (Named after Dutch physicist: Jan Burgers)

Vector of nonclosure of the Burger’s circuit

Assume positive dislocation line

\( \bigcirc \perp \mathbf{b} \)

FIG. 4.19 The Burgers circuit for an edge dislocation: (A) Perfect crystal and (B) crystal with dislocation counterclockwise

(1895~1981)
Procedure used to find the Burgers vectors of any dislocation

- Step by step Burgers circuit.
- $b$: connecting the end with the starting points.
- Convention: right hand circuit
• Arbitrary assumptions are involved in making a Burgers circuit.

• **An ambiguity** always exists in the direction of the Burgers vector.
Assume positive dislocation line

\[ \mathbf{b} \]: RHFS local Burgers vector

\[ \mathbf{b}' \]: true Burgers vector

(Occurs in a crystal where no distortion)

Imperfect crystal: edge dislocation
FIG. 4.20 The Burgers circuit for a dislocation in a screw orientation. (A) Perfect crystal and (B) crystal with dislocation.
• Summary:
(a) For a given dislocation, there is only one Burgers vector.
(b) A dislocation must end on itself (forming a loop) or on other dislocations, (forming a network), or on surface such as an external surface or a grain boundary.
(c) The slip (or glide) plane of a dislocation contains the dislocation line and its Burgers vector.
• Slip plane is the plane containing both the Burgers vector and the dislocation.

• The slip plane of an edge dislocation is uniquely defined

  ⇒ Because the Burgers vector \( \perp \) the dislocation line.
  ⇒ Edge dislocations are confined to move or glide in a unique plane.
4.4 Vector Notation for Dislocations

- In a real crystal, the arrangement of atoms is much more complex than previously mentioned cases
  => difficult to visualize the geometrical appearance of a complicated dislocation
  => dislocation is defined by its Burgers vector and dislocation line.
Closed-packed direction in simple cubic is \([100]\)

\(\Rightarrow\) the smallest shear (Burgers) vector = \([100]\)
4.5 Dislocations in the FCC Lattice

- Slip plane: \{111\}

\( b: \) Burgers vector
\( = \frac{1}{2} \langle 110 \rangle \)

**FIG. 4.22** A total dislocation (edge orientation) in a face-centered cubic lattice as viewed when looking down on the slip plane
• b: very large strain: each white atom forced to climb over the dark atom
• Alternative route: c + d: much smaller strain
Perfect and partial dislocations in FCC

Displacement of atoms by $b_1$ moves them to identical sites ⇒ glide of a **perfect dislocation** leaves perfect crystal structure.

Displacement of atoms by $b_2$ or $b_3$ is not a lattice vector ⇒ motion of **partial dislocation** leaves an imperfect crystal (stacking fault is created).
\[ b = \frac{1}{2}[-110] \]

\[ b = c + d \implies \]

\[ \frac{1}{2} [\overline{1} \overline{1} 0] = \frac{1}{6} [\overline{1} 2 \overline{1}] + \frac{1}{6} [\overline{2} 1 1] \]

\[ b: \text{ total dislocation} \]

\[ c, d: \text{ partial dislocation} \]

**FIG. 4.24** The orientation relationship between the Burgers vectors of a total dislocation and its partial dislocations
\[ \mathbf{b} = \mathbf{c} + \mathbf{d} \implies \]
\[ \frac{1}{2} [\overline{1}10] = \frac{1}{6} [\overline{1}21] + \frac{1}{6} [\overline{2}11] \]

- The strain energy of the system is lowered when a total dislocation breaks down into two partial dislocations.
  \implies \text{Dislocation energy is proportional to the square of Burgers vector (later in this chapter)}
  \implies \text{The break down is favored when } b^2 > c^2 + d^2.
• A repulsive force exists and forces the partials apart.
• Two equal partial dislocations (c & d): additional planes of atoms => a total dislocation => extended dislocation
The combination of the two partials is known as an extended dislocation.

two Shockley partials

\[
\frac{a}{6} [211] + \frac{a}{6} [12 \bar{1}]
\]
• If a stacking fault terminates inside a crystal, its boundaries will form a partial dislocation.
A stacking fault possesses a surface energy small compared with that of an ordinary grain boundaries.

Atoms inside a stacking fault are not at the positions they would normally occupy in a perfect crystal.

Stacking Fault Energy consideration:

1. Partial dislocations break down: energy lower
2. Partial dislocations interaction (repulsive) $\propto \frac{1}{d}$
3. Surface energy ($\gamma$) associated with stacking fault $\gamma \propto d$
The movement of an extended dislocation might be affected by
(1) **obstacles** (other dislocations, second-phase): the width of the stacking fault should vary.
(2) **thermal vibrations**: should cause the width to vary.
A, B: stacking faults

partial dislocations [5-2-3]

Stacking faults bordered by two dislocation lines.

**FIG. 4.26** An electron micrograph of a thin foil of lightly deformed Cu + 15.6 at. % Al alloy. The beam direction is close to [101] and \( g \) is indicated. X 44,000. (From A.K. Head, P. Humble, L.M. Clarebrough, A.J. Morton and C.T. Forwood, Computed Electron Micrographs and Defect Identification, American Elsevier Publishing Company, Inc., New York, 1973. Used by permission of the authors.)
4.6 Intrinsic and Extrinsic Stacking Faults in FCC metals (Frank type)

- Frank partial dislocations: intrinsic SF

$\Rightarrow$ The normal stacking sequence $ABCA_{\uparrow}C$ can be observed to exist right up to the plane of the fault. 
(the same as Shockley partials)

**FIG. 4.27A** An intrinsic stacking fault can also be formed in a face-centered cubic crystal by removing part of a close-packed plane

Difficult to move (glide) along $b$

Sessile dislocation

Climb only
Extrinsic or double stacking fault

=> stacking sequence ABCA↓C↓BC

=> By inserting part of a close-packed plane, not correctly stacked with respect to the planes on either side (up and down) of the fault.

**FIG. 4.27B** The addition of a portion of an extra close-packed plane to a face-centered cubic crystal produces an extrinsic stacking fault.
4.7 Extended Dislocations in Hexagonal Metals

- Close packed plane: basal plane: extended dislocations also occur in these metals
  => a total dislocation
  => dissociated partials

\[
\frac{1}{3} [\overline{1} 2 \overline{1} 0] = \frac{1}{3} [01 \overline{1} 0] + \frac{1}{3} [\overline{1} 100]
\]

FCC equivalent:

\[
\frac{1}{2} [\overline{1} 10] = \frac{1}{6} [\overline{1} 2 1] + \frac{1}{6} [211]
\]
4.8 Climb of Edge Dislocations:

- Slip plane: the plane contains both the dislocation line and its Burgers vector.
• Edge => only one possible plane
  \[\therefore \text{dislocation line} \perp \text{Burgers vector}\]
• There is another method, different from slip, by which an edge dislocation can move.

<table>
<thead>
<tr>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
</tr>
</thead>
</table>

![Diagram showing the process of positive climb of an edge dislocation](image)

**FIG. 4.29** Positive climb of an edge dislocation

• If an atom jumps into the vacancy, $\perp$ loses one atom. $\Rightarrow$ $\perp$ will climb one atomic distance perpendicular to the slip plane.
- If an atom jumps into the extra plane, $\perp$ gains one atom. 
  $\Rightarrow$ $\perp$ will climb one atomic distance perpendicular to the slip plane.
- Negative climb: resulting growth in size of the extra plane.
- Climb: not a cooperative movement of the entire row of atoms.
- Slip: a cooperative movement of the entire row of atoms.
Positive and negative climb of edge dislocations:

- **Shear stress** $\Rightarrow$ **slip** (on the slip plane)
- **Normal stress** (compressive & tensile) $\Rightarrow$ **climb** (⊥ the slip plane) (for an edge dislocation)
4.9 Dislocation Intersections

- On any given slip plane, there will be a certain number of dislocations that lie in this plane and capable of producing slip along it.

  => at the same time, there will be many other dislocations that intersect it at various angles
• One edge dislocation has moved across ABCD by $b$.
• The other is vertical dislocation loop.
• $\tau$ cannot break the loop into two separated half-loops.

**FIG. 4.31** In the figure a dislocation is assumed to have moved across the horizontal plane $ABCD$ and, in cutting through the vertical-dislocation loop, it forms a pair of steps in the latter

• Two basic types:
(1) kinks: steps lie in the slip plane of a dislocation

- Kinks or jogs: depending on how two dislocations interesting

Intersections of two edge dislocations

Edge dislocation $XY (\mathbf{b}_1)$

Edge dislocation $AB (\mathbf{b}_2)$

$\mathbf{b}_1 // \mathbf{b}_2$

D. Hull, D. J. Bacon Introduction to Dislocations, p. 137
- Kink in a edge dislocation is screw type.
- Kinks (on)

![Diagram of dislocations with kinks](image)

**Fig. 4.32** Dislocations with kinks that lie in the slip plane of the dislocations

- Step lies in the slip plane
- Both of steps can be eliminated easily (moving mn to dashed line).
Intersections of two edge dislocations

(2) jogs: steps are normal to the slip plane.

Edge dislocation XY \( \vec{b}_1 \)
Edge dislocation AB \( \vec{b}_2 \)

The length of PP’: \(|\vec{b}_1|\)

For the AB slip plane
• Jogs

mn, no, and op move over a stepped surface (not gliding along a single plane)

**FIG. 4.33** Dislocations with jogs normal to their slip planes
### Characteristics of Dislocations

<table>
<thead>
<tr>
<th>‖ Characteristics</th>
<th>Edge</th>
<th>Screw</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relation between t and b</td>
<td>perpendicular</td>
<td>parallel</td>
</tr>
<tr>
<td>Slip Direction</td>
<td>parallel to b</td>
<td>parallel to b</td>
</tr>
<tr>
<td>Change of glide plane</td>
<td>climb</td>
<td>cross-slip</td>
</tr>
<tr>
<td>Direction of ‖ motion relative to b</td>
<td>parallel</td>
<td>perpendicular</td>
</tr>
<tr>
<td>• Direction of ‖ motion relative to t</td>
<td>perpendicular</td>
<td>perpendicular</td>
</tr>
</tbody>
</table>
4.10 The Stress Field of a Screw Dislocation (nt/m²)

assume \( \vec{b} \) along \( \hat{z} \)

\[
\begin{align*}
  u_x &= 0 \\
  u_y &= 0 \\
  u_z &= \frac{b \theta}{2\pi} \left( \frac{b}{2\pi} = \frac{\theta}{\theta} \right) = \frac{b}{2\pi} \tan^{-1} \left( \frac{y}{x} \right)
\end{align*}
\]

\[
\theta = \tan^{-1} \left( \frac{y}{x} \right) \quad \frac{d}{dx} \tan^{-1} (u) = \frac{1}{1 + u^2} \frac{du}{dx}
\]
\[ \gamma_{xx} = \frac{du_x}{dx} = 0 \quad \gamma_{yy} = \frac{du_y}{dy} = 0 \quad \gamma_{zz} = \frac{du_z}{dz} = 0 \]

**strain**

\[ \gamma_{yz(zy)} = \frac{du_y}{dz} + \frac{du_z}{dy} = \frac{b}{2\pi} \frac{x}{x^2 + y^2} = \frac{b}{2\pi} \frac{r \cos \theta}{r^2} = \frac{b}{2\pi} \frac{\cos \theta}{r} \]

\[ \gamma_{zx(xz)} = \frac{du_z}{dx} + \frac{du_x}{dz} = -\frac{b}{2\pi} \frac{y}{x^2 + y^2} = -\frac{b}{2\pi} \frac{r \sin \theta}{r^2} = -\frac{b}{2\pi} \frac{\sin \theta}{r} \]

\[ \gamma_{xy(yx)} = \frac{du_x}{dy} + \frac{du_y}{dx} = 0 \]

\[ \tau = \mu \gamma \]

**stress**

\[ \tau_{yz} = \frac{\mu b}{2\pi} \frac{x}{x^2 + y^2} \quad \tau_{xz} = -\frac{\mu b}{2\pi} \frac{y}{x^2 + y^2} \]
\[ u_r = 0 \quad u_\theta = 0 \quad u_z = \frac{b \theta}{2\pi} \]

Shear strain

\[ r_{rr} = \frac{\partial u_r}{\partial r} = 0 \]
\[ r_{\theta\theta} = \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = 0 \]
\[ r_{zz} = \frac{\partial u_z}{\partial z} = 0 \]

\[ r_{r\theta} = \frac{\partial u_r}{r \partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = 0 \]
\[ r_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = 0 \]

\[ r_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{r \partial \theta} = \frac{\partial \left( \frac{b \theta}{2\pi} \right)}{r \partial \theta} = \frac{b}{2\pi r} \]

\[ \therefore \tau_{\theta z} = \mu r_{\theta z} = \frac{\mu b}{2\pi r} \]

(\(\mu\): shear modulus, nt/m$^2$)
4.10 The Stress Field of a Screw Dislocation (nt/m²)

- The analysis of the stresses close to the center of the dislocation is extremely difficult, and no completely satisfactory theory has yet been developed.
  - at the center, representation of the crystal as a homogeneous and isotropic medium becomes less and less realistic.
4.11 The Stress Field of an Edge Dislocation (nt/m²)

\[ \sigma_{xx} = \frac{-G|b|}{2\pi(1-\nu)} \frac{y(3x^2 + y^2)}{(x^2 + y^2)^2}; \quad \sigma_{yy} = \frac{G|b|}{2\pi(1-\nu)} \frac{y(x^2 - y^2)}{(x^2 + y^2)^2}; \]

\[ \tau_{xy} = \frac{G|b|}{2\pi(1-\nu)} \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}; \]

\[ \sigma_{rr} = \sigma_{\theta\theta} = \frac{-G|b|}{2\pi(1-\nu)} \left( \frac{\sin\theta}{r} \right); \quad \tau_{r\theta} = \frac{G|b|}{2\pi(1-\nu)} \left( \frac{\cos\theta}{r} \right); \]

; in Cartesian coordinate

; in polar coordinate
Assume an edge dislocation aligned along z-axis
Linear elasticity theory: six stresses and six strains

Stress and strain relationship:

\[ \sigma_{xx} = c_{11} \varepsilon_{xx} + c_{12} \varepsilon_{yy} + c_{13} \varepsilon_{zz} + c_{14} \varepsilon_{yz} + c_{15} \varepsilon_{zx} + c_{16} \varepsilon_{xy} \]

\[ \sigma_{yy} = c_{21} \varepsilon_{xx} + c_{22} \varepsilon_{yy} + c_{23} \varepsilon_{zz} + c_{24} \varepsilon_{yz} + c_{25} \varepsilon_{zx} + c_{26} \varepsilon_{xy} \]

\[ \sigma_{zz} = c_{31} \varepsilon_{xx} + c_{32} \varepsilon_{yy} + c_{33} \varepsilon_{zz} + c_{34} \varepsilon_{yz} + c_{35} \varepsilon_{zx} + c_{36} \varepsilon_{xy} \]

\[ \tau_{yz} = c_{41} \varepsilon_{xx} + c_{42} \varepsilon_{yy} + c_{43} \varepsilon_{zz} + c_{44} \varepsilon_{yz} + c_{45} \varepsilon_{zx} + c_{46} \varepsilon_{xy} \]

\[ \tau_{zx} = c_{51} \varepsilon_{xx} + c_{52} \varepsilon_{yy} + c_{53} \varepsilon_{zz} + c_{54} \varepsilon_{yz} + c_{55} \varepsilon_{zx} + c_{56} \varepsilon_{xy} \]

\[ \tau_{xy} = c_{61} \varepsilon_{xx} + c_{62} \varepsilon_{yy} + c_{63} \varepsilon_{zz} + c_{64} \varepsilon_{yz} + c_{65} \varepsilon_{zx} + c_{66} \varepsilon_{xy} \]
If we limit ourselves to isotropic materials.

\[
\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz}
\]

\[
\sigma_{yy} = \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda\varepsilon_{zz}
\]

\[
\sigma_{zz} = \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz}
\]

\[
\sigma_{yz} = \mu\varepsilon_{yz}
\]

\[
\sigma_{zx} = \mu\varepsilon_{zx}
\]

\[
\sigma_{xy} = \mu\varepsilon_{xy}
\]

\[
\begin{array}{cccccccc}
\varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \varepsilon_{yz} & \varepsilon_{zx} & \varepsilon_{xy} \\
\hline
\sigma_{xx} & \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\sigma_{yy} & \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\sigma_{zz} & \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
\sigma_{yz} & 0 & 0 & 0 & \mu & 0 & 0 \\
\sigma_{zx} & 0 & 0 & 0 & 0 & \mu & 0 \\
\sigma_{xy} & 0 & 0 & 0 & 0 & 0 & \mu \\
\end{array}
\]

\(\mu\) and \(\lambda\): Lame constant

(French mathematician)
According to Newton’s law: dynamic equilibrium for any materials

\[ \begin{align*}
    u : & \text{ displacement in the } x \\
    v : & \text{ displacement in the } y \\
    w : & \text{ displacement in the } z \\
\end{align*} \]

\[ \begin{align*}
    \rho \frac{\partial^2 u}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \\
    \rho \frac{\partial^2 v}{\partial t^2} &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\
    \rho \frac{\partial^2 w}{\partial t^2} &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \\
\end{align*} \]

\[ \rho \left( \frac{\text{kg}}{m^3} \right) : \text{ mass density} \]

\[ \sigma \left( \frac{\text{N/m}^2}{m^2} \right) : \text{ stress} \]
• For screw or edge dislocations (dislocation line // z)

\[
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 v}{\partial x \partial y}
\]

\[
\rho \frac{\partial^2 v}{\partial t^2} = \mu \frac{\partial^2 v}{\partial x^2} + (\lambda + 2\mu) \frac{\partial^2 v}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 u}{\partial x \partial y}
\]

\[
\rho \frac{\partial^2 w}{\partial t^2} = \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)
\]

\[
\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda \varepsilon_{yy} + \lambda \varepsilon_{zz}
\]
\[
\sigma_{yy} = \lambda \varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda \varepsilon_{zz}
\]
\[
\sigma_{zz} = \lambda \varepsilon_{xx} + \lambda \varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz}
\]
\[
\sigma_{yz} = \mu \varepsilon_{yz}
\]
\[
\sigma_{zx} = \mu \varepsilon_{zx}
\]
\[
\sigma_{xy} = \mu \varepsilon_{xy}
\]
\[ \rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \]

\[ = (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + \lambda \frac{\partial^2 v}{\partial x \partial y} + \lambda \frac{\partial^2 w}{\partial x \partial z} + \mu \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 w}{\partial x \partial z} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 w}{\partial x \partial z} + \mu \frac{\partial^2 u}{\partial z^2} = 0 \]

\[
\begin{align*}
\sigma_{xx} &= (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz} \\
\sigma_{yy} &= \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda\varepsilon_{zz} \\
\sigma_{zz} &= \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz} \\
\sigma_{yx} &= \mu\varepsilon_{yz} \\
\sigma_{zx} &= \mu\varepsilon_{zx} \\
\sigma_{xy} &= \mu\varepsilon_{xy}
\end{align*}
\]
• If the solution if \( u = v = 0, w = \frac{-b}{2\pi} \tan^{-1}\left(\frac{y}{x}\right) \)

\[\Rightarrow \text{Means a screw dislocation line along the z and displacement along z.}\]

• If the solution

\[u = \frac{-b}{2\pi} \left[ \tan^{-1}\left(\frac{y}{x}\right) + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{xy}{x^2 + y^2} \right]\]

\[v = \frac{-b}{2\pi} \left[ \frac{-\mu}{2(\lambda + 2\mu)} \ln \frac{x^2 + y^2}{c} + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{y^2}{x^2 + y^2} \right]\]

\[w = 0\]

\[\Rightarrow \text{means an edge dislocation.}\]
\[
\begin{align*}
    u &= \frac{-b}{2\pi} \left[ \tan^{-1} \left( \frac{y}{x} \right) + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{xy}{x^2 + y^2} \right] \\
    v &= \frac{-b}{2\pi} \left[ \frac{-\mu}{2(\lambda + 2\mu)} \ln \frac{x^2 + y^2}{c} + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{y^2}{x^2 + y^2} \right] \\
    w &= 0 \\
    \text{Poisson's ratio } \nu &= \frac{\lambda}{2(\lambda + \mu)} \\
    \text{strain} \\
    \varepsilon_{xx} &= \frac{by \mu y^2 + (2\lambda + 3\mu)x^2}{2\pi (\lambda + 2\mu)(x^2 + y^2)^2} \\
    \varepsilon_{yy} &= \frac{-by (2\lambda + \mu)x^2 - \mu y^2}{2\pi (\lambda + 2\mu)(x^2 + y^2)^2} \\
    \varepsilon_{xy} &= \frac{-b x(x^2 - y^2)}{2\pi(1-\nu) (x^2 + y^2)^2} \\
    \varepsilon_{zz} &= \varepsilon_{xz} = \varepsilon_{yz} = 0
\end{align*}
\]
\[ \varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left\{ \frac{-b}{2\pi} \left[ \tan^{-1}\left( \frac{y}{x} \right) + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{xy}{x^2 + y^2} \right] \right\} \]

\[ = \frac{by}{2\pi} \frac{\mu y^2 + (2\lambda + 3\mu)x^2}{(\lambda + 2\mu)(x^2 + y^2)^2} \]

\[ \varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{-by}{2\pi} \frac{(2\lambda + \mu)x^2 - \mu y^2}{(\lambda + 2\mu)(x^2 + y^2)^2} \]

\[ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{-b}{2\pi(1-\nu)} \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \]

\[ u = \frac{-b}{2\pi} \left[ \tan^{-1}\left( \frac{y}{x} \right) + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{xy}{x^2 + y^2} \right] \]

\[ v = \frac{-b}{2\pi} \left[ \frac{-\mu}{2(\lambda + 2\mu)} \ln \frac{x^2 + y^2}{c} + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{y^2}{x^2 + y^2} \right] \]

\[ w = 0 \]

\[ \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \]

\[ \varepsilon_{zz} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \]
\[
\sigma_{zz} = \lambda \varepsilon_{xx} + \lambda \varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz} \quad \text{example}
\]

\[
= \lambda \frac{by \mu y^2 + (2\lambda + 3\mu)x^2}{2\pi (\lambda + 2\mu)(x^2 + y^2)^2} - \lambda \frac{by(2\lambda + \mu)x^2 - \mu y^2}{2\pi (\lambda + 2\mu)(x^2 + y^2)^2}
\]

\[
= \frac{\lambda by}{\pi} \frac{\mu}{(\lambda + 2\mu)(x^2 + y^2)} = \frac{vby\mu}{\pi(1 - \nu)(x^2 + y^2)}
\]

\[
\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda \varepsilon_{yy} + \lambda \varepsilon_{zz}
\]
\[
\sigma_{yy} = \lambda \varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda \varepsilon_{zz}
\]
\[
\sigma_{zz} = \lambda \varepsilon_{xx} + \lambda \varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz}
\]
\[
\sigma_{yz} = \mu \varepsilon_{yz}
\]
\[
\sigma_{zx} = \mu \varepsilon_{zx}
\]
\[
\sigma_{xy} = \mu \varepsilon_{xy}
\]
Stress (from the book)

\[
\sigma_{xx} = \frac{\mu b}{2\pi(1-\nu)} \frac{y(3x^2 + y^2)}{(x^2 + y^2)^2}
\]

\[
\sigma_{yy} = \frac{-\mu b}{2\pi(1-\nu)} \frac{y(x^2 - y^2)}{(x^2 + y^2)^2}
\]

\[
\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = \frac{\mu b v y}{\pi(1-\nu)(x^2 + y^2)}
\]

\[
\sigma_{xy} = \frac{-\mu b}{2\pi(1-\nu)} \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}
\]

\[
\sigma_{xz} = \sigma_{yz} = 0
\]
4.11 The Stress Field of an Edge Dislocation:

\[ \sigma_x = \sigma_y = \frac{\mu b}{2\pi r (1-\nu)} \]

\[ \tau_{xy} = \frac{\mu b}{2\pi r (1-\nu)} \]

**FIG. 4.37** Stress and strain associated with an edge dislocation. (In the above equations, \( \mu \) is the shear modulus in Pa, \( b \) the Burgers vector of the dislocation, \( \nu \) is Poisson’s ratio, and \( r \) the distance from the center of the dislocation.)
From polar coordinates (in cylindrical coordinates)

From polar coordinates (in cylindrical coordinates)

\[ x^2 + y^2 = r^2 \]
\[ x = r \cos \theta \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \]
\[ y = r \sin \theta \]

The strain-displacement relations

\[ U_x = U_r \cos \theta - U_\theta \sin \theta \quad (1) \]
\[ U_y = U_r \sin \theta + U_\theta \cos \theta \quad (2) \]
\[ U_z = 0 \]
(u) 
\[ U_x = \frac{-b}{2\pi} \left[ \theta + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{xy}{x^2 + y^2} \right] = \frac{-b}{2\pi} \left[ \theta + \frac{1}{2(1 - \nu)} \sin \theta \cos \theta \right] \]

(v) 
\[ U_y = \frac{-b}{2\pi} \left[ \frac{-\mu}{2(\lambda + 2\mu)} \ln \frac{x^2 + y^2}{c} + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{y^2}{x^2 + y^2} \right] \]
\[ = \frac{-b}{2\pi} \left[ \frac{2\nu - 1}{4(1 - \nu)} \ln \frac{r^2}{c} + \frac{1}{2(1 - \nu)} \sin^2 \theta \right] \quad \text{Edge dislocation} \]

Poisson's ratio \( \nu = \frac{\lambda}{2(\lambda + \mu)} \) \quad 1 - \nu = \frac{\lambda + 2\mu}{2(\lambda + \mu)}
\[(1) \times \sin \theta \quad \& \quad (2) \times \cos \theta\]

\[
\sin \theta U_r \cos \theta - U_\theta \sin^2 \theta = \frac{-b}{2\pi} \theta \sin \theta - \frac{b}{4\pi(1-\nu)} \sin^2 \theta \cos \theta
\]

\[
\cos \theta U_r \sin \theta + U_\theta \cos^2 \theta = \frac{-b \cos \theta(2\nu - 1)}{8\pi(1-\nu)} \ln\left(\frac{r^2}{c}\right) - \frac{b}{4\pi(1-\nu)} \sin^2 \theta \cos \theta
\]

\[
-U_\theta (\sin^2 \theta + \cos^2 \theta) = \frac{-b \theta \sin \theta}{2\pi} + \frac{b \cos \theta(2\nu - 1)}{8\pi(1-\nu)} \ln\left(\frac{r^2}{c}\right)
\]

\[
U_\theta = \frac{b \theta \sin \theta}{2\pi} - \frac{b \cos \theta(2\nu - 1)}{8\pi(1-\nu)} \ln\left(\frac{r^2}{c}\right)
\]
\begin{align*}
\text{(1)} \ x \cos \theta & \quad & \text{(2)} \ x \sin \theta \\
U_x &= U_r \cos \theta - U_\theta \sin \theta \\
U_y &= U_r \sin \theta + U_\theta \cos \theta
\end{align*}

\[
\cos \theta U_r \cos \theta - U_\theta \sin \theta \cos \theta = \frac{-b\theta}{2\pi} \cos \theta - \frac{b}{4\pi (1 - \nu)} \sin \theta \cos^2 \theta
\]

\[
\sin \theta U_r \sin \theta + U_\theta \cos \theta \sin \theta = \frac{-b(2\nu - 1) \sin \theta}{8\pi (1 - \nu)} \ln\left(\frac{r^2}{c}\right) - \frac{b}{4\pi (1 - \nu)} \sin^2 \theta \sin \theta
\]

\[
U_r (\sin^2 \theta + \cos^2 \theta) = \frac{-b \theta \cos \theta}{2\pi} - \frac{b \sin \theta (2\nu - 1)}{8\pi (1 - \nu)} \ln\left(\frac{r^2}{c}\right)
\]

\[
- \frac{b \sin \theta}{4\pi (1 - \nu)} (\cos^2 \theta + \sin^2 \theta)
\]

\[
\text{strain} \quad r_{rr} = \frac{\partial U_r}{\partial r} = -\frac{b \sin \theta (2\nu - 1)}{8\pi (1 - \nu)} \frac{2r}{r^2} = -\frac{b (2\nu - 1) \sin \theta}{4\pi r (1 - \nu)}
\]
strain

\[ r_{\theta\theta} = \frac{\partial U_{\theta}}{r \partial \theta} + \frac{U_r}{r} = \frac{b}{2\pi r} (\sin \theta + \theta \cos \theta) + \frac{b(2\nu - 1)}{8\pi (1 - \nu) r} \ln\left(\frac{r^2}{c}\right) \sin \theta \]

\[ - \frac{b \theta \cos \theta}{2\pi r} - \frac{b \sin \theta (2\nu - 1)}{8\pi r (1 - \nu)} \ln\left(\frac{r^2}{c}\right) - \frac{b \sin \theta}{4\pi r (1 - \nu)} \]

\[ = \frac{b \sin \theta}{2\pi r} - \frac{b \sin \theta}{4\pi r (1 - \nu)} \]

\[ r_{zz} = 0 = \frac{\partial U_z}{\partial z} \]
\[ r_{r\theta} = \frac{\partial U_r}{r \partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} \]

\[ = -\frac{b}{2\pi r} (\cos \theta - \theta \sin \theta) - \frac{b(2\nu - 1)}{8\pi r(1 - \nu)} \ln\left(\frac{r^2}{c}\right) \cos \theta - \frac{b \cos \theta}{4\pi (1 - \nu) r} \]

\[ - \frac{b \cos \theta (2\nu - 1)}{8\pi r(1 - \nu)} \frac{2r}{r^2} - \frac{b \theta \sin \theta}{2\pi r} + \frac{b \cos \theta (2\nu - 1)}{8\pi r(1 - \nu)} \ln\left(\frac{r^2}{c}\right) \]

\[ = -\frac{b \cos \theta}{2\pi r(1 - \nu)} \]

\[ r_{rz} = r_{\theta z} = 0 \]
\[ \sigma_{rr} = (\lambda + 2\mu) \frac{b(2\nu - 1) \sin \theta}{4\pi r(1 - \nu)} + \frac{b\lambda \sin \theta}{2\pi r} - \frac{b\lambda \sin \theta}{4\pi r(1 - \nu)} \]
\[= \frac{b \sin \theta}{2\pi r(1 - \nu)} \left[ -\frac{1}{2} (\lambda + 2\mu)(2\nu - 1) + \lambda(1 - \nu) - \frac{1}{2} \lambda \right] \]
\[= \frac{\mu b}{2\pi(1 - \nu)} \frac{\sin \theta}{r} \]

\[\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda \varepsilon_{yy} + \lambda \varepsilon_{zz} \]
\[\sigma_{yy} = \lambda \varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda \varepsilon_{zz} \]
\[\sigma_{zz} = \lambda \varepsilon_{xx} + \lambda \varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz} \]
\[\sigma_{yz} = \mu \varepsilon_{yz} \]
\[\sigma_{zx} = \mu \varepsilon_{zx} \]
\[\sigma_{xy} = \mu \varepsilon_{xy} \]
stress

\[
\sigma_{\theta\theta} = \frac{-b\lambda(2\nu - 1)\sin\theta}{4\pi r(1 - \nu)} + \frac{(\lambda + 2\mu)b\sin\theta}{2\pi r} - \frac{(\lambda + 2\mu)b\sin\theta}{4\pi r(1 - \nu)} = \frac{\mu b}{2\pi(1 - \nu)} \frac{\sin\theta}{r}
\]

\[
\begin{align*}
\sigma_{xx} &= (\lambda + 2\mu)\varepsilon_{xx} + \lambda \varepsilon_{yy} + \lambda \varepsilon_{zz} \\
\sigma_{yy} &= \lambda \varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda \varepsilon_{zz} \\
\sigma_{zz} &= \lambda \varepsilon_{xx} + \lambda \varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz} \\
\sigma_{yz} &= \mu \varepsilon_{yz} \\
\sigma_{zx} &= \mu \varepsilon_{zx} \\
\sigma_{xy} &= \mu \varepsilon_{xy}
\end{align*}
\]
stress

\[ \sigma_{zz} = \lambda r_{rr} + \lambda r_{\theta \theta} + (\lambda + 2\mu) r_{zz} \]

\[ = \frac{-b\lambda (2\nu - 1) \sin \theta}{4\pi r(1 - \nu)} + \frac{\lambda b \sin \theta}{2\pi r} - \frac{\lambda b \sin \theta}{4\pi r(1 - \nu)} \]

\[ = \frac{b\lambda \sin \theta (1 - 2\nu)}{2\pi r (1 - \nu)} \quad (\therefore \nu = \frac{\lambda}{2(\lambda + \mu)}) \]

\[ = \frac{b\sin \theta}{\pi r (1 - \nu)} \cdot \frac{\lambda (1 - 2\nu)}{2} \quad (\therefore \lambda = \frac{2\mu \nu}{1 - 2\nu}) \]

\[ = \frac{\mu \nu b}{\pi (1 - \nu)} \cdot \frac{\sin \theta}{r} \]

\[ \sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda \varepsilon_{yy} + \lambda \varepsilon_{zz} \]

\[ \sigma_{yy} = \lambda \varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda \varepsilon_{zz} \]

\[ \sigma_{zz} = \lambda \varepsilon_{xx} + \lambda \varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz} \]

\[ \sigma_{yz} = \mu \varepsilon_{yz} \]

\[ \sigma_{zx} = \mu \varepsilon_{zx} \]

\[ \sigma_{xy} = \mu \varepsilon_{xy} \]
4.13 Strain Energy of a screw dislocations (J/m)

- Summation of elastic energy of the stress field.

Total strain energy:
\[ S = \frac{1}{2} \times \text{stress} \times \text{strain} \]

For a generalized stress field, the strain energy density (or stored energy per volume) \( E \) (J/m³)

\[ r = E = \int \sigma dr = \int \mu r dr \]

\( (nt/m^2) \)
\( (J/m^3) \)

For a generalized stress field, the strain energy density (or stored energy per volume) \( E \) (J/m³)

\[ = \frac{1}{2} \left( \sigma_{xx} r_{xx} + \sigma_{yy} r_{yy} + \sigma_{zz} r_{zz} + \sigma_{xy} r_{xy} + \sigma_{yz} r_{yz} + \sigma_{xz} r_{xz} \right) \]
• If in terms of strains alone \((r_{xx}, r_{yy}, r_{zz} \ldots)\)

\[
E = \frac{1}{2} (\lambda + 2\mu)(r_{xx} + r_{yy} + r_{zz})^2 + \frac{1}{2}\mu(r_{xy}^2 + r_{xz}^2 + r_{yz}^2) - 4r_{yy}r_{zz} - 4r_{xx}r_{zz} - 4r_{xx}r_{yy})
\]

\[
\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz}
\]
\[
\sigma_{yy} = \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda\varepsilon_{zz}
\]
\[
\sigma_{zz} = \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz}
\]
\[
\sigma_{yz} = \mu\varepsilon_{yz}
\]
\[
\sigma_{zx} = \mu\varepsilon_{zx}
\]
\[
\sigma_{xy} = \mu\varepsilon_{xy}
\]

• If in terms of stresses alone \((\sigma_{xx}, \sigma_{yy}, \sigma_{zz} \ldots)\)

\[
E = \frac{1}{2\mu}\left[\frac{\lambda + \mu}{3\lambda + 2\mu}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) + (\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)\right] - \frac{\lambda}{3\lambda + 2\mu}(\sigma_{xx}\sigma_{zz} + \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz})
\]
For a screw dislocation \((\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0)\)

\[
\tau_{yz} = \frac{\mu b}{2\pi} \frac{x}{x^2 + y^2} \quad \tau_{xz} = \frac{-\mu b}{2\pi} \frac{y}{x^2 + y^2}
\]

\[
E = \frac{1}{2\mu} (\tau_{xz}^2 + \tau_{yz}^2)
\]

\[
= \frac{1}{2\mu} \left( \frac{\mu b}{2\pi} \right)^2 \frac{1}{x^2 + y^2} = \frac{1}{2\mu} \left( \frac{\mu b}{2\pi} \right)^2 \frac{1}{r^2} \left( \frac{J}{m^3} \right)
\]

\[
\therefore \text{strain energy (J/m) per unit length}
\]

\[
W_s = \int_{r_0}^{r'} \frac{1}{2\mu} \left( \frac{\mu b}{2\pi} \right)^2 \frac{1}{r^2} 2\pi r dr = \int_{r_0}^{r'} \left( \frac{\mu b}{2\pi r} \right)^2 \left( \frac{1}{2\mu} \right) 2\pi r dr
\]

\[
= \frac{\mu b^2}{4\pi} \ln\left(\frac{r'}{r_0}\right) \left( \frac{J}{m} \right)
\]
When \( y = 0 \)

\[
\sigma_{yz} = \frac{\mu b}{2\pi} \frac{x}{x^2} = \frac{\mu b}{2\pi} \frac{1}{x}
\]

\[
\therefore -\int_0^b \sigma_{yz} \, db = \frac{\mu}{2\pi x} \int_0^b b \, db = \frac{1}{2} \frac{\mu b^2}{2\pi x}
\]

\[
\therefore W_s = \frac{\mu b^2}{4\pi} \int_{r_0}^{r'} dx = \frac{\mu b^2}{4\pi} \ln \frac{r'}{r_0}
\]
4.14 Strain Energy of an edge dislocations (J/m)

For an edge dislocation \((\sigma_{xz} = \sigma_{yz} = 0)\)

(1) Method one: obvious that a great deal of tedious calculation is involved.

From stresses alone \((\tau_{xz} = \tau_{yz} = 0)\)

\[
E = \frac{1}{2\mu} \left[ \frac{\lambda + \mu}{3\lambda + 2\mu} \left( \sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 \right) + \left( \tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2 \right) \right] \\
- \frac{\lambda}{3\lambda + 2\mu} \left( \sigma_{xx}\sigma_{zz} + \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} \right)
\]
\[ \sigma_{xx} = \frac{\mu b}{2 \pi (1 - \nu)} \frac{y(3x^2 + y^2)}{(x^2 + y^2)^2} \]

\[ \sigma_{yy} = \frac{-\mu b}{2 \pi (1 - \nu)} \frac{y(x^2 - y^2)}{(x^2 + y^2)^2} \]

\[ \sigma_{zz} = \nu (\sigma_{xx} + \sigma_{yy}) = \frac{\mu \nu by}{\pi (1 - \nu)(x^2 + y^2)} = \frac{\mu b 2 \nu y(x^2 + y^2)}{2 \pi (1 - \nu) (x^2 + y^2)^2} \]

\[ \sigma_{xy} = \frac{-\mu b}{2 \pi (1 - \nu)} \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \]

\[ \sigma_{xz} = \sigma_{yz} = 0 \]

Poisson's ratio \( \nu = \frac{\lambda}{2(\lambda + \mu)} \)

\[ 1 - \nu = \frac{\lambda + 2\mu}{2(\lambda + \mu)} \]
\[
E = \frac{1}{2\mu} \frac{1}{(x^2 + y^2)^2} (x^2 + y^2 - \frac{\mu}{\lambda + \mu}) \frac{\mu^2 b^2}{4\pi(1 - \nu)^2} 
\therefore D = \frac{\mu b}{2\pi(1 - \nu)}
\]

\[
= \frac{\mu b^2}{8\pi^2(1 - \nu)(x^2 + y^2)} = \frac{\mu b^2}{8\pi^2(1 - \nu)r^2} \left( \frac{J}{m^3} \right)
\]

\[
\therefore W_s = \int_{r_0}^{r'} \frac{\mu b^2}{8\pi^2(1 - \nu)r^2} 2\pi r dr
\]

\[
= \frac{\mu b^2}{4\pi(1 - \nu)} \int_{r_0}^{r'} \frac{dr}{r} = \frac{\mu b^2}{4\pi(1 - \nu)} \ln \frac{r'}{r_0}
\]

if \( r_0 = \frac{b}{4} \)

\[
W_s = \frac{\mu b^2}{4\pi(1 - \nu)} \ln \frac{4r'}{b}
\]
• For an edge dislocation, the energy is modified by Poisson’s ratio \( \nu \Rightarrow 1/(1-\nu) \).

• For most metals, \( \nu \) is about 1/3, so the strain energy for the edge dislocation is about 50% larger than that for the screw dislocation. (because \( x^{3/2} \))
4.12 Force on a screw dislocation (nt/m): glide

- The work done by shear stress $\tau$ which causes the crystal to slip by a distance $b$ is

$$W_\tau = (\text{force}) \times (\text{distance}) = \tau (L \Delta x) \ b \ (\text{external work})$$

$$= \tau \times b \times L \times \Delta x$$

$$= f \ (\text{virtual force per unit length}) \times L \times \Delta x$$
4.12 Force on an edge dislocation (nt/m): glide

- \( f = \tau \times b \)
- Force acts normal to the dislocations lines for both types of dislocations.
- Force normal to the dislocation everywhere around the loop.

\[ \sum_x = \sigma_{xx} b_x + \tau_{xy} b_y + \tau_{xz} b_z; \]
\[ \sum_y = \tau_{yx} b_x + \sigma_{yy} b_y + \tau_{yz} b_z; \]
\[ \sum_z = \tau_{zx} b_x + \tau_{zy} b_y + \sigma_{zz} b_z. \]

**FIG. 4.40** A positive edge dislocation
4.12 Force on an edge dislocation (nt/m): climb

- The climb forces on an edge dislocations: similar argument
  $\Rightarrow f = \sigma b$

$\sigma$ : normal stress;

$\perp$ dislocation line
$\perp$ slip plane as well.
(different from screw and edge dislocations)
• In the general case, dislocations are neither pure screw nor pure edge, and the dislocation line may lie in any direction. => common practice to define the orientation of the dislocation line by the unit vector tangent to the dislocation line at the point.

=> the dislocation line vector \((\xi(zi): \xi_x, \xi_y, \xi_z)\), Burgers vector \(b = b_x i + b_y j + b_z k\)

=> A general vector \((\Sigma: \Sigma_x, \Sigma_y, \Sigma_z)\), where

\[
\Sigma_i = \sum_{j=1}^{3} \sigma_{ij} b_j
\]

\[
\begin{align*}
\Sigma_x &= \sigma_{xx} b_x + \tau_{xy} b_y + \tau_{xz} b_z ; \quad \text{(on the x plane)} \\
\Sigma_y &= \tau_{yx} b_x + \sigma_{yy} b_y + \tau_{yz} b_z ; \quad \text{(on the y plane)} \\
\Sigma_z &= \tau_{zx} b_x + \tau_{zy} b_y + \sigma_{zz} b_z \quad \text{(on the z plane)}
\end{align*}
\]

• \(\Sigma\) ay have both normal and shear components in all directions
• The force \((f)\) on the dislocation is \(f = \sum \times \xi.\)

\[
\begin{vmatrix}
i & j & k \\
\Sigma_x & \Sigma_y & \Sigma_z \\
\xi_x & \xi_y & \xi_z
\end{vmatrix}
\]

• \(f = \sum x \xi = (b \cdot G) \times \xi\)

\(f \perp \sum\) and dislocation line

(known as the Peach-Kohler equation: 1950)